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1989 J. Phys. A: Math. Gen. 22 2871

(http://iopscience.iop.org/0305-4470/22/14/034)

J. Phys. A: Math. Gen. **22** (1989) 2871-2875. Printed in the UK

COMMENT

Multifurcations and sequence-dependent universal constants

Viktor Urumov

Institut za Fizika, Univerzitet 'Kiril i Metodij' p. fah 162, Skopje 91001, Yugoslavia

Received 23 February 1989

Abstract. An infinite number of forward and reverse n-furcations of tangent bifurcations of unimodal maps can be obtained from a simple substitutional rule. Besides the usual Feigenbaum-type universal constants, new numbers emerge that are independent of the particular map, but for a given n depend on the type of sequence under consideration.

Non-linear phenomena represent a subject of increased interest in recent years because of their extensive presence in different fields of natural sciences and due to successes both in the theoretical and experimental efforts of their study. As a simplest model, which is also immensely fruitful, scientists frequently turn to one-dimensional maps $x_{i+1} = f_a(x_i)$ of finite interval. Such maps exhibit many universal characteristics which are also typical in complex systems of higher dimensionality. Two collections of representative papers in the field were published recently, almost simultaneously (Cvitanović 1984, Hao 1984). In the following we draw attention to some new, apparently universal, constants with the novel feature of their dependence on the particular sequence under examination.

Consider the logistic mapping defined by

$$f_a(x) = 1 - ax^2 \qquad |x| \leqslant 1 \qquad 0 \leqslant a \leqslant 2 \tag{1}$$

where $f_a(x)$ has a quadratic maximum at $x = 0$. Depending on the value of the control parameter a, the iteration $x_{i+1} = f_a(x_i)$ displays periodic or chaotic behaviour. In the domains of periodic behaviour, the increase of a gives rise to the well known period-doubling process (Feigenbaum 1978, 1979) characterised by universal constants. If $\{\bar{a}_{p2^m}\}$ describes a sequence with the property that at each $\bar{a}_{p2^m}$, a $p2^m$ cycle bifurcates to a $p2^{m+1}$ cycle, the universal constant δ satisfies

$$\delta = \frac{\bar{a}_{p2^{m+1}} - \bar{a}_{p2^m}}{\bar{a}_{p2^{m+2}} - \bar{a}_{p2^{m+1}}} = 4.669\,201\,606. \tag{2}$$

The same limiting law holds for the so-called superstable values $a_s = a_{p2^m}$ defined by the condition

$$f_{a_s}^{(p2^m)}(0) = 0 \tag{3}$$

i.e. starting at $x = 0$ after $p2^m$ applications of the mapping $f_{a_s}(x)$ one returns to the origin. Similar behaviour was found (Derrida *et al* 1979) in the ordering of periodic windows borne by tangent bifurcations (Pomeau and Manneville 1980). The n-furcation sequences of superstable values a_{pn^m} for cycles of period pn^m with $m = 0, 1, 2, \ldots,$ for large m obey the asymptotic relationship

$$a_{pn^m} = a_\infty(f, p, n) - \frac{A(f, p, n)}{\delta^m(n)}. \tag{4}$$

This is an equivalent form of the statement (2). Here the limit $a_\infty(f, p, n)$ and the amplitude $A(f, p, n)$ depend on the particular sequence and on the function defining the mapping, whereas δ is a universal constant determined only by the parameter n that defines the n-furcation process.

The orbits belonging to the n-furcation sequence can be specified in terms of the $*$-composition law (Derrida *et al* 1978) and the MSS sequences (Metropolis *et al* 1973). Each superstable orbit in symbolic dynamics of MSS sequences is represented by a word which consists of letters L and R depending on whether successive iterates fall to the left or right of the critical point $x_c = 0$. The initial and final position x_c, sometimes denoted by C (for centre) is usually understood and omitted. The order in which the MSS sequences appear with the increase of the tuning parameter is universal, independent of the specific unimodal function defining the iteration process. For two admissible words $P = P_1P_2 \dots P_{p-1}$ and $Q = Q_1Q_2 \dots Q_{n-1}$, where P_i, $Q_j = \mathrm{L}, \mathrm{R}$; $i = 1, 2, \dots, p-1$, $j = 1, 2, \dots, n-1$, the $*$-composition law is defined by

$$P * Q = P\tau_1 P\tau_2 P \dots P\tau_{n-1} P \tag{5}$$

where $\tau_i = Q_i$ if P is an even sequence and $\tau \neq Q_i$ in the opposite case. The sequence P is even (odd) if it contains an even (odd) number of R characters. The n-furcation sequences mentioned previously are obtained from $P * Q^{*n}$, where Q^{*n}, $Q * Q * \dots * Q$ means $n-1$ applications of the composition law ($P * Q^{*n} = P$ for $n = 0$). All such sequences follow the direction of increase of the control parameter.

In the following we consider the sequences defined by $Q^{*m} * P$, $m = 0, 1, 2, \dots$, which represent superstable cycles of period pn^m. As an example, in the case $P = \mathrm{RL}$ and $Q = \mathrm{R}$, the first few terms in the sequence are

$$\mathrm{RL},\ \mathrm{RLR^3},\ \mathrm{RLR^3LRLRLR},\ \mathrm{RLR^3LRLRLR^3LR^3LR^3LR}, \dots . \tag{6}$$

From this example it becomes evident that each subsequent term in the sequence can be obtained from the preceding one by the substitution

$$\begin{aligned} \mathrm{R} &\to \mathrm{RL} \\ \mathrm{L} &\to \mathrm{RR} \end{aligned} \tag{7}$$

and addition of an R at the end. Similar substitutional or inflation rules were considered by Procaccia *et al* (1987). In the more general case of n-furcations, the rules in which the $Q * P$ composition law can be rephrased, read as follows: each R and L from P is replaced by

$$\left.\begin{aligned} \mathrm{R} &\to Q\mathrm{R} \\ \mathrm{L} &\to Q\mathrm{L} \end{aligned}\right\} \text{if } Q \text{ is even} \qquad \text{and} \qquad \left.\begin{aligned} \mathrm{R} &\to Q\mathrm{L} \\ \mathrm{L} &\to Q\mathrm{R} \end{aligned}\right\} \text{if } Q \text{ is odd} \tag{8}$$

and the sequence Q is added at the end. In other terms, R (L) is substituted by an odd (even) sequence starting with Q, while at the end one adds another Q.

It is known (Metropolis *et al* 1973, Derrida *et al* 1978) that for two arbitrary and different sequences A and B an ordering can be established. One says that

$$A < B \tag{9}$$

if the corresponding superstable parameters satisfy

$$a < b. \tag{10}$$

Defining the order $\mathrm{L} < \mathrm{C} < \mathrm{R}$ on the letters, the criterion of Derrida *et al* (1978) determines the order between A and B as follows: there is always a least integer k for which $A_k \neq B_k$, then $A < B$ if and only if $A_k < B_k$ and $A_1A_2 \ldots A_{k-1}$ is even or $A_k > B_k$ and $A_1A_2 \ldots A_{k-1}$ is odd.

It is an elementary exercise, having (9) and using the reinterpretation of the composition law, to verify that

$$Q * A < Q * B \tag{11}$$

for any Q. For example if Q and $A_1A_2 \ldots A_{k-1}$ are even and $A_k = \mathrm{L}$, $B_k = \mathrm{C}$ and therefore $A_k < B_k$ and $A < B$, the substitution rule for the first terms in A and B which differ from each other, gives $Q\mathrm{L} < Q\mathrm{C}$ and correspondingly $Q * A < Q * B$. Similarly one can examine all other possible cases with the same result, thus establishing the order-preservation property of the composition law.

All the sequences $Q^{*m} * P$ tend to the same limiting value $a_\infty(Q)$ independently of the initial sequence P. This follows from the fact that all sequences $(Q^{*m}) * P$ when $m \to \infty$ have an increasing coincident part at the beginning which tends to an infinitely long word $Q^{*\infty}$. On the other hand the preservation of the order shows that if the initial sequence P is to the right (left) of $Q^{*\infty}$ the sequence $Q^{*m} * P$ approaches its limit from right (left). Thus for the superstable values of the cycles $Q^{*m} * P$ in the limit of large m we may write

$$a_m(f, P, Q) = a_\infty(f, Q) \mp \frac{A(f, P, Q)}{\delta^m(Q)} \tag{12}$$

where $-$ and $+$ stand for forward and reverse (or backward) sequences. The limiting value $a_\infty(f, Q)$ depends on Q which defines the n-furcations and on the particular map $f_a(x)$, but not on the initial sequence, whereas the universal constant δ is a function of Q only.

Investigation of sequences of this type was performed by Delbourgo and Kenny (1985) who found that they are governed by the same $\delta(Q)$ as the forward n-furcation sequences $P * Q^{*m}$. They also found that there are different universal functions which arise from the n-furcation process $Q^{*m} * P$ that satisfy the same renormalisation group equation of Feigenbaum-Cvitanović. Their study is limited to the reverse sequences which do not exhaust all the possible cases.

In connection with such sequences we present numerical evidence that the ratio

$$\frac{A(f, P, Q)}{A(f, Q, Q)} = r(P, Q) \tag{13}$$

is a universal constant in the sense that it is independent of the map $f_a(x)$ and which depends only on the type of n-furcation process defined by Q and on the initial sequence P. Here the fundamental forward sequence Q^{*m} is used to set the scale $A(f, Q, Q)$ with which the other amplitudes $A(f, P, Q)$ are compared. Similar universal numbers, defined as ratios of amplitudes, were previously reported by Lorentz (1980) and Lutzky (1988).

The results are displayed in table 1. The calculations were performed with two maps that are not conjugate to each other ($f_a(x) = 1 - ax^2$ and $g_{a'}(x) = a' \cos(\pi x) - 0.5$). In the numerical procedure we have used the method of Kaplan (1983) which directly determines the superstable orbits of desired type. After the initial calculations on a pocket programmable calculator, the numerical analysis was carried out with double precision in which the superstable parameters were found with an accuracy of 10^{-15} and the number of bi- and trifurcations was as high as 12 while for the quintufurcations

Table 1. The universal constant r for forward and reverse sequences of a given type. The accumulation points $a_\infty(a'_\infty)$ correspond to the functions $f(x) = 1 - ax^2$ ($g(x) = a'\cos(\pi x) - 0.5$). The three possible period-five sequences RLR^2, RL^2R, RL^3 are denoted respectively by $5a$, $5b$, $5c$. The last column gives the exponent δ for the multifurcation process (Chang and McCown 1984).

Cycle sequence	Initial MSS sequence	Sequence defining the multifurcation	r	$a_\infty(a'_\infty)$	δ
$2^n \times 5a$	RLR^2	R	0.496 568	1.401 155 189	4.6692
$2^n \times 3$	RL	R	0.802 632	(0.865 578 869)	
$2^n \times 5b$	RL^2R	R	1.067 96		
$2^n \times 4$	RL^2	R	1.272 36		
$2^n \times 5c$	RL^3	R	1.413 48		
2×3^n	R	RL	23.693	1.786 440 255	55.247
$5a \times 3^n$	RLR^2	RL	4.924 6	(0.946 975 311)	
$3^n \times 5b$	RL^2R	RL	2.272 6		
$3^n \times 4$	RL^2	RL	4.765 0		
$3^n \times 5c$	RL^3	RL	6.235 1		
$5a^n \times 5b$	RL^2R	RLR^2	35.029	1.631 926 654	255.55
$5a^n \times 5c$	RL^3	RLR^2	53.989		
$5a \times 5b^n$	RLR^2	RL^2R	163.43	1.862 224 022	1 287.1
$5b^n \times 5c$	RL^3	RL^2R	85.365		
$5a \times 5c^n$	RLR^2	RL^3	3070	1.985 539 530	16 931
$5b \times 5c^n$	RL^2R	RL^3	1060		

we limited ourselves to 6 steps. It is evident from the data that the constant r increases with the distance of the starting sequence P from the accumulation point.

In conclusion, the forward and reverse n-furcation sequences $Q^{*m} * P$ that were considered, have unique accumulation point that depends on Q. Further, the ratio of amplitudes is apparently another universal constant which is independent of the particular mapping. One may suppose that similar properties characterise unimodal maps with non-quadratic behaviour $|x|^z$ in the vicinity of the critical point, in which case r becomes a function of z. Two obvious questions remain. Are there any other substitutional rules which lead to some n-furcation sequences? How one can extract the constant r from the renormalisation group equation?

Acknowledgment

I am grateful to Professor Delbourgo for sending me his paper and to the Institute of Earthquake Engineering and Seismology in Skopje for the free access to their computing facilities.

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